American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS)
ISSN (Print) 2313-4410, ISSN (Online) 2313-4402

http://asrjetsjournal.org/

# Several Chaotic Approaches of One Dimensional Doubling Map

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**Abstract**

In this paper, we study basic dynamical behavior of one-dimensional Doubling map. Especially emphasis is given on the chaotic behaviors of the said map. Several approaches of chaotic behaviors by some pioneers it is found that the Doubling map is chaotic in different senses. We mainly focused on Orbit Analysis, Sensitivity to Initial Conditions, Sensitivity to Numerical Inaccuracies, Trajectories and Staircase Diagram of the Doubling map. The graphical representations show that this map is chaotic in different senses. The behavior of the said map is found irregular, that is, chaotic.

***Keywords:*** Approaches; Orbit; Sensitivity; Staircase Diagram; Trajectories; Transitivity.

## 1. Introduction

Dynamical Systems is a branch of mathematics that attempts to understand processes in motion. There are many branches of Dynamical Systems but Chaos is one of them and it explains how very small changes in the initial configuration of a system model may lead great discrepancies over time. Called 'Butterfly Effect', this phenomena accounts our accurate prediction for a long period of time. A dictionary definition of Chaos is a '*disordered state of collection; a confused mixture*.' With the Advancement of chaos research, more and more chaos phenomena have been discovered in the mathematics, engineering and other fields. Many researchers have focused on the demonstration of chaos phenomena using simulation.

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The graphical analysis of the Doubling map significantly has contributed to our understanding of the essence of chaos. The use of the term "Chaos" was first introduced into dynamical systems by Li and Yorke [7] for a map on a compact interval. Another explicit definition of chaos belongs to Devaney [11]. Then Robinson gave a refined definition [1]. The remarkable feature of the Doubling map is in the simplicity of its form and the complexity of its dynamics. It is the simplest model that shows chaos.

The Doubling map

$$D(x)=\begin{cases}2x & \text{if } 0\le x<0.5\\ 2x-1 & \text{if } 0.5\le x<1.\end{cases}$$

is very interesting and representative model of dynamical systems. The dynamical behavior of the Doubling map is very complicated. We have shown that Doubling map is chaotic in the sense of **Devaney, Li-Yorke, Lyapunov and Wiggin.** In this paper, we discussed chaotic dynamical behavior of one dimensional Doubling map. The Doubling map is topologically conjugate to the Logistic map Shift map and also semi-conjugacy to Tent map. We also have shown that the dynamical behavior of Doubling map is chaotic considering the *Initial Seeds, Orbit Analysis, Sensitivity to Initial Conditions, Sensitivity to Numerical Inaccuracies, Trajectories and Staircase Diagram.* These are the main results of this article.

## 2. Methodology

There are some of senses to analyze the chaos like Henri $\text{Poincare}'$, R. L. Devaney, Li and Yorke, Lyapunov Exponent, Wiggins etc. In our research we tried to present one dimensional Doubling map in the sense of R. L. Devaney, Li and Yorke, Lyapunov Exponent, Wiggins etc. We use mathematical software's like Mathematica, MATLAB to analyse the numerical results to be found in the future so that we can describe the graphical representation of our mathematical research.

## 3. Mathematical Preliminaries

We need some basic definitions to be used elsewhere of this article. We mention these definitions as follows.

### *3.1 Basic Definitions*

#### *3.1.1 Orbit*

Given $x_0 \in R$ ($x_0$ is called the seed or initial value of the orbit), we define the orbit of $x_0$ under $f$ to be the sequence $\left\{x_0,\ x_1 = f(x_0),\ x_2 = f^2(x_0),\ldots,\ x_n = f^n(x_0),\ldots\right\}$.

For example, Let $f(x)=\sqrt{x}$ and $x_0=16$, then the orbit of $x_0$ under $f$ to be the sequence $\{16,4,2,1.414,\ldots\}$.

We see that the point on this orbit tend to 1.

#### *3.1.2 Periodic Orbit or Cycle*

The point $x_0$ is called periodic if $f^n(x_0) = x_0$ for some $n > 0$, where $n$ is called the prime period of the orbit.

#### *3.1.3 Periodic Orbit with Prime Period n*

If the point $x_0$ is periodic with prime period $n$, then

the periodic orbit with prime period $n$ to be the sequence

$$\{x_0, f(x_0), \ldots, f^{n-1}(x_0), x_0, f(x_0), \ldots, f^{n-1}(x_0), \ldots\}.$$

For example, let $f(x) = -x^3$ and $x_0 = 1$, then $f(1) = -1$, $f^2(1) = f(f(1)) = 1$. Thus 1 is periodic point with prime period 2. Similarly, -1 is periodic point with prime period 2 and these orbits are: $\{1, -1, 1, -1, \ldots\}$ and $\{-1, 1, -1, 1, \ldots\}$.

#### *3.1.4 Transitivity*

A dynamical system is transitive if for any pair of points x and y and any $\varepsilon > 0$ there is a third point z within $\varepsilon$ of x whose orbit comes within $\varepsilon$ of y.

In other words, a transitive dynamical system has the property that, given any two points, we can find an orbit that comes arbitrarily close to both.

#### *3.1.5 Sensitivity to Initial Conditions*

A dynamical system $f$ depends sensitively on initial conditions if there is a $\beta > 0$ such that for any *x* and any $\varepsilon > 0$ there is a *y* within $\varepsilon$ of *x* and a *k* such that the distance between $f^k(x)$ and $f^k(y)$ is at least $\beta$. In this definition it is important to understand the order of the quantifiers. The definition says that, no matter which *x* we begin with and no matter how small a region we choose about *x*. we can always find a *y* in this region whose orbit eventually separates form that of *x* by at least $\beta$. The distance $\beta$ is independent of *x*. as a consequence, for each *x*, there are points arbitrarily nearby whose orbits are eventually "far" that of *x*. The idea of sensitive dependence on initial conditions is very important topics in the study of dynamical system. If a particular system possesses sensitive dependence, then for all practical purpose, the dynamics of this system defy numerical computation.

Mathematically, A continuous map $f : X \to X$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood $N(x)$ of $x$, there exist $y \in N(x)$, $n \geq 0$ such that $d\left(f^n(x), f^n(y)\right) > \delta,$ where $(X, d)$ is a compact metric space.

### *3.1.6 Topologically Transitivity*

Consider the metric space $X$ and the continuous map $f : X \to X$ .We say that $f$ is topological transitive if for every pair of non-empty open sets $U$ and $V$ in $X$ there exists a positive integer $k$ such that $f^k(U) \cap V \neq \phi$ .

### *3.1.7 Topological Conjugacy*

Let $f : A \to A$ and $g : B \to B$ be two continuous mappings. Then $f$ and $g$ are said to be topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $h \circ f = g \circ h$ . The homeomorphism $h$ is called a topological conjugacy between $f$ and $g$ .

***3.1.9 Compact Space:*** Let $X$ and $Y$ be metric spaces. We say that the metric space $X$ is compact if every open cover of $X$ has a finite sub cover, i.e. if $\{I_i\}_{i \in I}$ is a collection of open sets of $X$ such that $X \subset \bigcup_{i \in I} Ii$ then we have that $X \subset \bigcup_{i=1}^{n} I_i$ .Also compact spaces on the real line can be thought as closed and bounded intervals.

### *3.1.10 Lyapunov Exponent*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and differentiable map. Then $\forall x \in \mathbb{R}$ we define the (local) Lyapunov exponent of $x$ say $\lambda(x)$ as $\lambda(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left| f'(x_i) \right| \ \forall X_i \in \mathbb{R}$ .

### *3.1.11 Scrambled Set*

Consider an interval $I$ and the continuous map $f : I \to I$ .Then an uncountable subset $S$ of $I$ containing no periodic points of $f$ is said to be scrambled if:

1. Any Periodic point $p$ of $f$ and any point $x \in I$ satisfies $\lim_{n \to \infty} \sup\left| f^n(x) - f^n(p) \right| > 0$ .

2. $\forall$x,y $\in X$, $\lim_{n \to \infty} \sup\left| f^n(x) - f^n(y) \right| > 0$ and $\lim_{n \to \infty} \inf\left| f^n(x) - f^n(y) \right| = 0.$

### *3.2 Several Definitions of Chaos*

In the following we give some important mathematical definitions of chaos by some pioneers**.**

***3.2.1 Devaney's Definition [9] (R. L. Devaney 1989)***

Let $X$ be a metric space. A continuous function $f : X \to X$ is said to be *chaotic* on $X$ if f has the following three properties:

(C-1) Periodic points are dense in the space $X$

(C-2) *f* is topologically transitive

(C-3) *f has* sensitive dependence on initial conditions

Mathematically,

(C-1) $P_k(f) = \{x \in X : f^k(x) = x (\exists k \in \mathbf{N})\}$ is dense in $X$.

(C-2) For $\forall\ U, V : non-empty\,open\,sets$ of X, $\exists k \in \mathbf{N}$ such that $f^k(U) \cap V \neq \phi.$

(C-3) $\exists \delta > 0$ (Sensitive constant) which satisfies: $\forall\ x \in X\ \ and\ \ \forall N(x, \varepsilon),\ \exists\, y \in N(x, \varepsilon)$ and $\exists\ k \leq 0$ such that $d(f^k(x), f^k(y)) > \delta$ $[4]$.

In other words, a continuous map $f : X \to X$ on a compact metric space $X$ is called chaotic in the sense of Devaney – or just D-chaotic if there exists a compact invariant subset $Y$ (called a D-chaotic set ) of $X$ with the following properties:

$(i)$ $f\,|\,Y$ is transitive, $(ii)$ $P(f\,|\,Y) = Y$ , $(iii)$ $f\,|\,Y$ has sensitive dependence on initial conditions.

***3.2.2 Li and Yorke Definition [7]***

A continuous map $f : X \to X$ on a compact metric space $(X,\ d)$ is called chaotic in the sense of *Li and Yorke* – or just L/Y-chaotic – if there exists an uncountable subset $S$ (called a scrambled set ) of $X$ with the following properties:

$(i)$ $\lim\limits_{n\to\infty} sup\ d\left(f^n(x),\ f\ n\ (y)\right) > 0$ for all $x,\ y \in S,\ x \neq y,$

$(ii)$ $\lim\limits_{n\to\infty}$ inf $d\left(f^n(x),\ f^n(y)\right) = 0$ for all $x,\ y \in S,\ x \neq y,$

$(iii)$ $\lim\limits_{n\to\infty}$ sup $d\left(f^n(x),\ f^n(y)\right)$ > 0 for all $x \in S,\ p \in X, p$ periodic.

*In other words,* let $I$ be an interval and let $f : I \to I$ be a continuous map with a periodic point of period three. Then $f$ is said to be chaotic in the sense of *Li-Yorke or L-Y* chaotic if $f$ has an uncountable scrambled set.

### *3.2.3 Lyapunov Definition [1]*

Consider the continuous and differentiable map $f : \mathbb{R} \to \mathbb{R}$ .Then $f$ is said to be chaotic according to Lyapunov or L-chaotic if:

1. $f$ is topologically transitive.

2. $f$ has a positive Lyapunov exponent.

### *3.2.4 Wiggin's Definition [13]*

Let $f : X \to X$ be a continuous map and $X$ be a metric space .Then the map $f$ is said to be chaotic according to Wiggins or W-chaotic if:

1. $f$ is topologically transitive.

2. $f$ exhibits sensitive dependence on initial conditions.

## 4. Theorems and Propositions

### *4.1 Theorem*

*(Period Three implies Chaos). Suppose $F : \mathbb{R} \to \mathbb{R}$ is continuous. If $F$ has a periodic point of prime period 3, then $F$ has periodic points of all other periods.*

***Proof.*** First we need to established two lemma that follows from the continuity of $F$ .

Fixed Point Lemma: Suppose $I$ and $J$ are closed intervals such that $I \subset J$ .If $J \subset F(I)$ then F has fixed point in $I$

Pre-image Lemma: Suppose $I$ and $J$ are closed intervals such that $J \subset F(I)$. Then there exists a closed subinterval $I' \subset I$ such that $J = F(I')$. Let $F$ has a 3-cycle given by $a \to b \to c \to a$. We will assume $a < b < c$, the other cases are handled similarly. Let $I_0 = [a,b]$ and $I_1 = [b,c]$. Then we have $I_1 \subset F(I_0)$ and $I_0 \cup I_1 \subset F(I_1)$.First we note that since $I_1 \subset F(I_1)$, F has a fixed point in $I_1$, by the fixed point lemma. Now we will find a period 2-cycle. First we have that $I_1 \subset F(I_0)$, so by the pre-image lemma, there exists a subset $A_0 \subset I_0$ such that $I_1 = F(A_0)$. On the other hand $I_0 \subset F(I_1)$, so in fact $I_0 \subset F^2(A_0)$. Then by the fixed point lemma, there is a fixed point for $F^2$ in $I_0$. So we have a point of period 2 for F. In fact, since the iteration of this point leaves $I_0$ it cannot be a fixed point for $F$ , so it has prime period 2. We will find a periodic cycle of period n for all *n>3* by involving the Pre-image Lemma *n* times. Since $I_1 \subset F(I_1)$, there is a closed subinterval $A_1 \subset I_1$ such that $I_1 = F(A_1)$ . Now again we have $A_1 \subset F(A_1)$ , so there is a closed interval $A_2 \subset A_1$ such that $A_1 = F(A_2)$. Thus $I_1 = F^2(A_2)$. Continue this process for *n-2* steps to produce the following nested collection of closed subintervals: $A_{n-2} \subset A_{n-3} \subset ... \subset A_2 \subset A_1 \subset I_1$, such that $A_i = F(A_i + 1)$ and $I_1 = F^{n-2}(A_{n-2})$. Now let's bring in $I_0$ . We have $A_{n-2} \subset I_1 \subset F(I_0)$ , So there is a closed subintervals $A_{n-1} \subset I_0$ such that $A_{n-2} = F(A_{n-1})$. Finally we also have $A_{n-1} \subset I_0 \subset F(I_1)$, so there is a closed subinterval $A_n \subset I_1$ such that $A_{n-1} = F(A_n)$. What we have accomplished is the following: $A_n \xrightarrow{F} A_{n-1} \xrightarrow{F} ... \xrightarrow{F} A_1 \xrightarrow{F} I_1$

Where $A_n \subset I_1$ and $I_1 = F^n(A_n)$. But by the fixed point lemma, this means that there is a fixed point for $F_n$ in $A_n$ .Lets call this fixed point $x_0$. Then $x_0$ is a periodic point of period $n$ for $F$ . In fact, since the first iterate of $x_0$ lies in $I_1$, while the next $n-1$ iterates lie in $I_1$ we know that $x_0$ has prime period.

***4.2 Proposition***

*The doubling map* $D(x) = \begin{cases} 2x & \text{for } 0 \le x < \frac{1}{2} \\ 2x-1 & \text{for } \frac{1}{2} \le x < 1 \end{cases}$ *is chaotic on* $[0,1)$ *in the sense of Devaney.*

***Proof.*** First, we check that $D$ has a dense set of periodic points.

We claim that every rational number $x = \frac{p}{q}$ with $q$ odd is a periodic point for $D$ .This follows by observing that $D$ is a bijection from the set $\left\{\frac{1}{q}, \frac{2}{q}, ..., \frac{q-1}{q}\right\}$ to itself: clearly $D$ maps this set into itself, and it is also surjective because $\frac{2k}{q} = D\left(\frac{k}{q}\right)$ and $\frac{2k-1}{q} = D\left(\frac{k+(q-1)/2}{q}\right)$ But the rational numbers with odd denominator are dense in $[0,1)$ so $D$ has a dense set of periodic points.

Second, we claim that $D$ has a dense orbit, so (in particular) it is transitive. To do this, let $\alpha = 0.\underbrace{01}_{length1}\underbrace{00011011}_{length2}\underbrace{000001...111}_{length3}...$ be the base-2 decimal constructed by listing all sequences of length 1, then all sequences of length 2, then all sequences of length 3, and so forth. Note that, in base $2, D(x)$ is obtained simply by deleting the first digit of the base-2 decimal expansion of $\alpha$ (i.e., it acts essentially as the shift map).So in particular, for any sequence of digits, there is a shift of $\alpha$ that begins with that sequence of digits. Now let $x = 0.d_1d_2d_3...$ and $\varepsilon > 0$. We will show there is some shift of $\alpha$ within $\varepsilon$ of *x*. Choose *n* with $2^{-n} < \varepsilon$ .

Then there is a positive integer $k$ such that $D^k(\alpha)$ begins as $0.d_1 d_2 \dots d_n d_{n+1}$, so that $D^k(\alpha)$ and $x$ can only differ past the $n + 2nd$ decimal place. Then $\left|D^k(\alpha) - x\right| \le \sum_{i=n+2}^{\infty} \frac{2}{2^i} < 2^{-n} < \varepsilon$ as required.

Finally, we show that $D$ has sensitive dependence.

We will show that the value $\beta = \frac{1}{3}$ will satisfy the requirements of the definition.

First, observe that if $a, b$ are both in $\left[0, \frac{1}{2}\right)$ or $\left[\frac{1}{2}, 1\right)$, then $\left|D(b) - D(a)\right| = 2\left|b - a\right|$.

Also, if $a \in \left[0, \frac{1}{2}\right)$ and $b \in \left[\frac{1}{2}, 1\right)$ then one of $\left|b - a\right|$ and $\left|D(b) - D(a)\right|$ is at least $\frac{1}{3}$, since if $b - a < \frac{1}{3}$ then $\left|D(b) - D(a)\right| = \left|1 - 2(b - a)\right|$ is larger than $\frac{1}{3}$.Therefore, if $x$, $y$ are any two distinct points, the value of $\left|D^n(y) - D^n(x)\right|$ will double at each stage until the points $x$ and $y$ land in opposite halves of $[0,1)$,at which point either $\left|D^n(y) - D^n(x)\right|$ will exceed $\frac{1}{3}$ or $\left|D^{n+1}(y) - D^{n+1}(x)\right|$ will.

Thus, for any two distinct points $x$ and $y$, their orbits will eventually be a distance of at least apart after iterating some number of times [3] .

#### *4.3 Proposition*

*The Doubling map* $D(x) = \begin{cases} 2x & \text{for } 0 \le x < \frac{1}{2} \\ 2x - 1 & \text{for } \frac{1}{2} \le x < 1 \end{cases}$ *is chaotic on* $[0,1)$ *in the sense of Lyapunov exponent.*

***Proof.*** The give Doubling map is $D(x) = \begin{cases} 2x\,; & 0 \le x < 0.5 \\ 2x - 1\,; & 0.5 \le x < 1 \end{cases}$

In this case, $\left|D'(x)\right| = 2 > 1$, so the Doubling map is not attracted to a sink and this is not asymptotically periodic. It is easy to compute the Lyapunov Exponent as

$$h_1(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left|D'(x)\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln 2 = \ln 2 > 0$$

Since each orbit of $D(x)$ is asymptotically periodic and Lyapunov Exponent $h_1(x) > 0$, then theDoublingmap is Chaotic on [0,1) [1].

#### *4.4 Proposition*

*The Doubling map* $D(x)=\begin{cases}2x\,; & 0\le x<0.5\\ 2x-1\,; & 0.5\le \mathrm{x}<1\end{cases}$ *is chaotic with prime period three or Li and York Sense.*

We know the doubling map is $D(x)=\begin{cases}2x\,; & 0\le x<0.5\\ 2x-1\,; & 0.5\le \mathrm{x}<1\end{cases}$

Let the initial seeds $x_0=\frac{1}{3},\frac{1}{5},\frac{1}{6},\frac{1}{9},\frac{1}{7},\frac{2}{13},\frac{11}{24},.........$

When $x_0=\frac{1}{3};\ (i)\quad \left\{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},.........\right\}$, When $x_0=\frac{1}{5};\ (ii)\quad \left\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{1}{5},\frac{3}{5},\frac{2}{5},.........\right\}$

When $x_0=\frac{1}{6};\ (iii)\quad \left\{\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},.........\right\}$, When $x_0=\frac{1}{9};\ (iv)\quad \left\{\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{8}{9},\frac{7}{9},\frac{5}{9},\frac{1}{9},\frac{2}{9},.........\right\}$

When $x_0=\frac{1}{7};$ $\quad(v)\quad \left\{\frac{1}{7},\frac{2}{7},\frac{4}{7},\frac{1}{7},\frac{2}{7},\frac{4}{7},\frac{1}{7},\frac{2}{7},.........\right\}$ , When $x_0=\frac{1}{11};$

$(vi)\quad \left\{\frac{1}{11},\frac{2}{11},\frac{4}{11},\frac{8}{11},\frac{5}{11},\frac{10}{11},\frac{9}{11},\frac{7}{11},\frac{3}{11},\frac{6}{11},\frac{1}{11},.........\right\}$

We see that *(i), (ii), (iii), (iv), (vi)* have the period 2, 4, 2, 6, 9 respectively but (v) has prime period 3. So the Doubling Map is chaotic in the sense of Li-Yorke [7].

***4.5 Proposition***

*Doubling Map is chaotic in the sense of Wiggin.*

***Proof.*** Consider the Doubling Map $D(x):[0,1)\to[0,1)$ given by $D(x)=2x\bmod 1=\begin{cases}2x & ;\ 0\le x<0.5\\ 2x-1 & ;\ 0,5\le x\le 1\end{cases}$

First I will prove that $D(x)$ is transitive using symbolic dynamics .We let $\Sigma$ be the metric space of all infinite sequences containing 0's and 1's equipped with the metric $\rho(s,\tau)=\frac{1}{2^i}\left|s_i-\tau_i\right|\ \forall s=(s_0s_1s_2.......)$ and $\tau=(\tau_0\tau_1\tau_2......)\in\Sigma$ and we define $\sigma:\Sigma\to\Sigma$ given by $\sigma(s_0s_1s_2........)=(s_1s_2s_3........)$. Then there exist a point $x=(0100011011000001....)$ created by blocks of $0's$ and $1's$ which has a dense orbit. So $\sigma$ is transitive and then $B(x)$ is transitive [13]. Now I will prove that $B(x)$ has a dense set of periodic points. We have that $\mathrm{Fix}(B)=Per_1(B)=\{0\}\Rightarrow\left|Per(f)\right|=1=2^1-1.$ The second iterated map $B^2$ is given by $B^2(x)=4x\bmod 1$ and $Per_2(B)=\left\{0,\frac{1}{3},\frac{2}{3}\right\}\Rightarrow\left|Per_2(B)\right|=3=2^2-1.$ Generalizing this result the n-th iterated map is given by $B^n(x) = 2^n x \bmod 1$. So $Per_n(B) = \left\{0, \frac{1}{2^n-1}, \frac{2}{2^n-1}, ......\frac{2^n-2}{2^n-1}\right\}$ and $|Per_n(B)| = 2^n - 1$.Now $\lim_{n\to\infty} |Per_n(B)| = \infty$ so $\forall x \in [0,1)$ and $\forall \varepsilon > 0, N_\varepsilon(x)$ will contains a periodic point. Hence the periodic points of $B$ are dense. Since all the conditions for the three chaotic definitions are satisfied, so the map $D(x)$ is W-chaotic [7].Note: This map can be stated as chaotic map in the sense of J. Banks etc. al. because of Transitivity + Density implies Sensitivity [6].

***4.6 Proposition***

*Let $D$ is defined by* $D(x) = \begin{cases} 2x & if \quad 0 \le x < 0.5 \\ 2x-1 & if \quad 0.5 \le x < 1 \end{cases}$ $D(x)$ *is conjugate to Logistic Map.*

Let $h$ be the homeomorphism defined by $h(x) = Sin^2 2\pi x$ and we put $\varphi(x) = \left(h \circ D \circ h^{-1}\right)(x)$.

Then we have,

$$\begin{aligned}
\varphi(x) &= h\left(D\left(h^{-1}(x)\right)\right) \\
&= h\left(D\left(\frac{1}{2\pi} Sin^{-1}\sqrt{x}\right)\right), \; 0 \le \frac{arcSin\sqrt{x}}{2\pi} < 1 \quad (i.e. 0 \le x < 1) \\
&= h\left(2 \times \frac{1}{2\pi} Sin^{-1}\sqrt{x}\right), \; 0 \le \frac{arcSin\sqrt{x}}{2\pi} < \frac{1}{2} \quad \left(i.e. 0 \le x < \frac{1}{2}\right) \\
&= h\left(\frac{1}{\pi} Sin^{-1}\sqrt{x}\right), \; 0 \le \frac{arcSin\sqrt{x}}{2\pi} < \frac{1}{2} \quad \left(i.e. 0 \le x < \frac{1}{2}\right) \\
&= \left\{Sin\left(2\pi \times \frac{1}{\pi} Sin^{-1}\sqrt{x}\right)\right\}^2 \\
&= \left\{Sin\left(2Sin^{-1}\sqrt{x}\right)\right\}^2 \\
&= \left\{2Sin\left(Sin^{-1}\sqrt{x}\right).Cos\left(Sin^{-1}\sqrt{x}\right)\right\}^2 \\
&= \left(2\sqrt{x}\right)^2 \left\{1 - Sin^2\left(Sin^{-1}\sqrt{x}\right)\right\} \\
&= 4x(1-x)
\end{aligned}$$

And in the other case we have,

$$\varphi(x) = h\left(D\left(h^{-1}(x)\right)\right)$$
$$= h\left(D\left(\frac{1}{2\pi}Sin^{-1}\sqrt{x}\right)\right),\ 0 \le \frac{arcSin\sqrt{x}}{2\pi} < 1 \quad (i.e. 0 \le x < 1)$$
$$= h\left(2\times\frac{1}{2\pi}Sin^{-1}\sqrt{x} - 1\right),\ 0.5 \le \frac{arcSin\sqrt{x}}{2\pi} < 1 \quad (i.e. 0 \le x < 1)$$
$$= h\left(\frac{1}{\pi}Sin^{-1}\sqrt{x} - 1\right),\ 0.5 \le \frac{arcSin\sqrt{x}}{2\pi} < 1 \quad \left(i.e. 0 \le x < \frac{1}{2}\right)$$

$$= \left\{Sin 2\pi\left(\frac{1}{\pi}Sin^{-1}\sqrt{x} - 1\right)\right\}^2$$
$$= \left\{Sin\left(2Sin^{-1}\sqrt{x} - 2\pi\right)\right\}^2$$

$$= \left\{Sin\left(2Sin^{-1}\sqrt{x}\right)\right\}^2$$
$$= \left\{2Sin\left(Sin^{-1}\sqrt{x}\right).Cos\left(Sin^{-1}\sqrt{x}\right)\right\}^2$$
$$= \left(2\sqrt{x}\right)^2\left\{1 - Sin^2\left(Sin^{-1}\sqrt{x}\right)\right\}$$
$$= 4x(1-x)$$

Hence we have shown that *D* is chaotic map [9].

***4.7 Proposition***

*Let $D$ is defined by* $D(x) = \begin{cases} 2x & if \quad 0 \le x < 0.5 \\ 2x-1 & if \quad 0.5 \le x < 1 \end{cases}$ $D(x)$ *is Semi-Conjugacy with Tent map.*

We will show that the doubling map is semi-conjugate to the tent map *T* via *T* itself! That is, we will show that

$$\begin{array}{ccc} [0,1] & \xrightarrow{D} & [0,1] \\ T\downarrow & & \downarrow T \\ [0,1] & \xrightarrow{T} & [0,1] \end{array}$$

Thus *T* will be chaotic. This is because orbits under iteration of *D* map to dynamically equivalent orbits under *T*. In fact we now prove by induction that

$$T \circ D^{n-1} = T^n \qquad (1)$$

For all $n > 0$. Suppose Equation-1 is true for $n := k$. Then $T \circ D^{k-1} = T^k \Rightarrow T \circ T \circ D^{k-1} = T \circ T^k$

And since $T \circ D = T \circ T$, we have $T \circ D^k = T^{k+1}$

which completes the inductive proof. We remark that (1) gives an explicit formula for $T^n(x)$ since we already know that $D^{n-1}(x)=2^{n-1}x \bmod 1$. We now show that D is semi conjugate to $T$ via $T$, or in other words, that $T \circ D = T \circ T$. There four cases to consider for $T \circ T$.

$$0 \le x \le 1/4 \;\Rightarrow\; 0 \le T(x) \le 1/2 \;\Rightarrow T \circ T(x) = T(2x) = 2(2x) = 4x$$

$$1/4 \le x \le 1/2 \;\Rightarrow\; 1/2 \le T(x) \le 1 \;\Rightarrow T \circ T(x) = T(2x) = 2-2(2x) = 2-4x$$

$$1/2 \le x \le 3/4 \;\Rightarrow\; 1/2 \le T(x) \le 1 \;\Rightarrow T \circ T(x) = T(2-2x) = 2-2(2-2x) = 4x-2$$

$$3/4 \le x \le 1 \;\Rightarrow\; 0 \le T(x) \le 1/2 \;\Rightarrow T \circ T(x) = T(2-2x) = 2(2-2x) = 4-4x$$

Similarly, there are four cases for $T \circ D$:

$$0 \le x \le 1/4 \;\Rightarrow\; 0 \le D(x) \le 1/2 \;\Rightarrow T \circ D(x) = T(2x) = 2(2x) = 4x$$

$$1/4 \le x \le 1/2 \;\Rightarrow\; 1/2 \le D(x) \le 1 \;\Rightarrow T \circ D(x) = T(2x) = 2-2(2x) = 2-4x$$

$$1/2 \le x \le 3/4 \;\Rightarrow\; 0 \le D(x) \le 1/2 \;\Rightarrow T \circ D(x) = T(2x-1) = 2(2x-1) = 4x-2$$

$$3/4 \le x \le 1 \;\Rightarrow\; 1/2 \le D(x) \le 1 \;\Rightarrow T \circ D(x) = T(2x-1) = 2-2(2x-1) = 4-4x$$

We have to be a little bit careful at x=1/2 since D is not continuous there, and also at x=1 since we have not yet defined *D(1)*. But the reader may check that $T \circ D(1/2) = T \circ T(1/2) = 0$, and that $T \circ D(1) = T \circ T(1) = 0$ provided we defined $D(1)$ to be either 0 or 1. It is also straight forward to check that both $T \circ T$ and $T \circ D$ are continuous on $[0,1]$. So what we have shown is that

$$T \circ D(x) = T \circ T(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/4 \\ 2-4x & \text{if } 1/4 \le x \le 1/2 \\ 4x-2 & \text{if } 1/2 \le x \le 3/4 \\ 4-4x & \text{if } 3/4 \le x \le 1 \end{cases}$$

And so $D$ is conjugate to $T$ via $T$ for the graph of $T \circ T = T \circ D$. [9]

***4.8 Proposition***

*Doubling Map* $D(\mathrm{x}) = \begin{cases} 2x & ;\ 0 \le x < \dfrac{1}{2} \\ 2x-1 & ;\ \dfrac{1}{2} \le x < 1 \end{cases}$ *is Conjugate to Shift Map.*

***Proof.*** The doubling Map is defined by $D(\mathrm{x}) = \begin{cases} 2x & ;\ 0 \le x < \dfrac{1}{2} \\ 2x-1 & ;\ \dfrac{1}{2} \le x < 1 \end{cases}$

Let's represent each $x \in [0,1)$ by its binary expansion: $x = 0._2 b_1 b_2 b_3 ... = \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + ...$ where each $b_i \in \{0,1\}$. For $x = \frac{1}{2^n}$, represent $x$ with a binary expansion ending in 0's, rather than 1's. Then if $b_1 = 0$, we know $x \in [0,\ 1/2)$ Similarly, $b_1 = 1$, we know $x \in [1/2,1)$. Suppose $x \in [0,\ 1/2)$ is given. $x = 0._2 0 b_2 b_3 b_4 ... = \frac{0}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + ...$ .Then $D(x) = 2x = 0._2 0 b_2 b_3 b_4 ... = \frac{0}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} + ...$

Consider $x = 0._2 01010110.....$Then $x = 1/4 + 1/16 + 1/64 + 1/128 + \ ...$ and $D(x) = 2x = 2/4 + 2/16 + 2/64 + 2/128 = 1/2 + 1/8 + 1/32 + 1/64 + ...\ 1/64...\ = 0._2 1010110....$

Now suppose $x \in [1/2,1)$ is given. Then $x = 0._2 1 b_2 b_3 b_4 ... = \frac{1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \frac{b_4}{2^4} ....$

Then $D(x) = 2x - 1 = 1._2 b_2 b_3 b_4 ... -1 = 0._2 b_2 b_3 b_4 ...$ Hence on $[0,1)$, $D(x)$ is equivalent to $\sigma(x)$, the shift map on two symbols. We have already seen that the shift map $\sigma(x)$ is chaotic on the entire space of sequences of two symbols; hence *D(x)* is chaotic on the entire interval [0, 1) [chaos notes].

***4.9 Proposition***

*Let* $N_f(x)$ *be Newton iteration function associated to* $f(x) = x^2 + 1$. *Then* $N_f(x)$ *is conjugate to* $D(x)$.

***Proof.*** Let $D(x) = \begin{cases} 2x & 0 \le x < 1/2 \\ 2x-1 & 1/2 \le x < 1 \end{cases}$ be chaotic on [0,1), and let $N_f(x) = x - \dfrac{x^2+1}{2x} = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$ be defined on $\mathbb{R}$ Define the conjugacy function $h:[0,1) \to \mathbb{R}$ by $h(x) = \cot(\pi x)$. Then we have

$$h \circ D(x) = \cot(\pi \cdot D(x)) = \cot(2\pi x) = \frac{\cos^2(\pi x) - \sin^2(\pi x)}{2\sin(\pi x)\cos(\pi x)} = \frac{1}{2}(\cot(\pi x) - \tan(\pi x)) = N_f \circ h(x).$$

Therefore, $N_f(x)$ is conjugate to $D(x)$. That is, **w**e have the following commutative diagram:

$$\begin{tikzcd} {[0,1)} \arrow[r, "D"] \arrow[d, "h"'] & {[0,1)} \arrow[d, "h"] \\ \text{i} \arrow[r, "N_f"'] & \text{i} \end{tikzcd}$$

Thus the proof is complete [9].

***4.10 Proposition***

*The doubling function $D$ is chaotic on the unit circle.*

***Proof.*** We will again make use of a semi-conjugacy. Define the function $B: C^1 \to [-2,2]$ be given by $B(\theta) = 2\cos(\theta)$. Since $\cos(\theta)$ is the *x*-coordinate of the point $\theta$ on $C^1$, the map $B$ is given geometrically by projecting points vertically from the circle o the *x*-axis, and then stretching by a factor of 2 ( Fig. 10.5)…

Note that $B$ is two-to-one except at the points $\pi$ and 0 on $C^1$.Consider the diagram

$$\begin{tikzcd} C^1 \arrow[r, "D"] \arrow[d, "B"'] & C^1 \arrow[d, "B"] \\ {[-2,2]} \arrow[r, "?"'] & {[-2,2]} \end{tikzcd}$$

As before we ask which function completes the diagram. We have, $B \circ D(\theta) = 2\cos(\theta)$

So we must find the function that takes, $2\cos(\theta) \mapsto 2\cos(2\theta)$

However, we may write, $2\cos(2\theta) = 2(2\cos^2(\theta) - 1) = (2\cos(\theta))^2 - 2 = Q_{-2}(x)$

Then the required diagram is

$$\begin{tikzcd} C^1 \arrow[r, "D"] \arrow[d, "B"'] & C^1 \arrow[d, "B"] \\ {[-2,2]} \arrow[r, "Q_{-2}(x)"'] & {[-2,2]} \end{tikzcd}$$

So the required function is our friend the quadratic function $Q_c(x) = x^2 - 2$ . Thus $D$ and $Q_{-2}$ are semi conjugate. It is not difficult to mimic the arguments given above to the complete the proof [10].

## 5. Results and Discussion: Dynamical Behavior of Doubling Map

### *5.1 Chaotic Behavior of Doubling Map Considering Initial Seeds*

The Doubling Map is given by $D(\mathrm{x}) = \begin{cases} 2x & ;\ 0 \le x < \dfrac{1}{2} \\ 2x-1 & ;\ \dfrac{1}{2} \le x < 1 \end{cases}$

Considering the initial seeds $x_0 = 0.3, x_0 = 0.7, x_0 = \frac{1}{8}, x_0 = \frac{1}{16}, x_0 = \frac{1}{7}, x_0 = \frac{1}{14}, x_0 = \frac{1}{11}, x_0 = \frac{3}{22}.$

For each of the following seeds discuss the behavior of the resulting orbits under D(x).

(i) $x_0 = 0.3$ Since $D(0.3) = 0.6, D(0.6) = 0.2, D(0.2) = 0.4, D(0.4) = 0.8$ and $D(0.8) = 0.6$

The orbit of $0.3$ is eventually periodic with period 1 and period 4. We write $0.3 \in \mathrm{per}_4^1 D$

(ii) $x_0 = 0.7$ . Since $D(0.7) = 0.4,$ and since $0.4 \in \mathrm{per}_4 D$ from exercise before it follows that $0.7 \in \mathrm{per}_4^1 D$ .

(iii) $x_0 = \frac{1}{8}$; $\frac{1}{8}$ a $\frac{2}{8}$ a $\frac{4}{8}$ a $1 \bmod 1 = 0$ .But 0 is fixed by $D$ .Therefore $\frac{1}{8} \in \mathrm{per}_1^3 D \subseteq \mathrm{fix}\ D.$

(iv) $x_0 = \frac{1}{16}$; $\frac{1}{16}$ a $\frac{2}{16}$ a $\frac{4}{16}$ a $\frac{8}{16}$ a $0.$ Therefore, $\frac{1}{16} \in \mathrm{per}_1^4 D$ .

(v) $x_0 = \frac{1}{7}$; Since $D\left(\frac{1}{7}\right) = \frac{2}{7}, D\left(\frac{2}{7}\right) = \frac{4}{7},$ and $D\left(\frac{4}{7}\right) = \frac{1}{7},$ we have that $\frac{1}{7} \in \mathrm{per}_3 D$ .

(vi) $x_0 = \frac{1}{14}$; Since $D\left(\frac{1}{14}\right) = \frac{2}{14} = \frac{1}{7}$. But it was shown in (e) that $\frac{1}{7} \in \mathrm{per}_3 D$ .Therefore, $\frac{1}{14} \in \mathrm{per}_3^1 D$ .

(vii) $x_0 = \frac{1}{11}$; $\frac{1}{11}$ a $\frac{2}{11}$ a $\frac{4}{11}$ a $\frac{8}{11}$ a $\frac{5}{11}$ a $\frac{10}{11}$ a $\frac{9}{11}$ a $\frac{7}{11}$ a $\frac{3}{11}$ a $\frac{6}{11}$ a $\frac{1}{11}$, we see that $\frac{1}{11} \in \mathrm{per}_{10} D$ .

(viii) $x_0 = \frac{3}{22}$ ;since $D\left(\frac{3}{22}\right) = \frac{6}{22} = \frac{3}{11}\ per_{10} \in D$ ,we have that $\frac{3}{22} \in \mathrm{per}_{10}^1 D$

But we can see that for initial seed $x_0 = \frac{1}{7}$, this is a periodic point with prime period-3 and therefore the Doubling Map is Chaotic by the Period Three Theorem [7].

***5.2 Orbit Analysis of Doubling Map by Newton's Iteration***

The behavior of orbit of Newton iteration function associated to Doubling function

The doubling function $D(x)=\begin{cases} 2x & 0\le x<1/2 \\ 2x-1 & 1/2\le x<1 \end{cases}$ is chaotic on $[0,1]$. Obviously, $D$ has two roots 0 and $1/2$ which are on the real line.

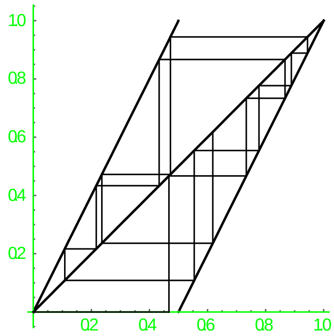



**Figure 1:** The graph of doubling function

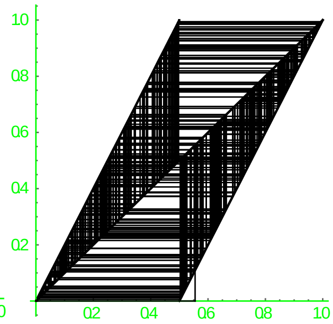



**Figure 2:** Chaos in doubling function

(1) The Newton iteration function associated to $D(x)$ is $N(x)=x-\dfrac{2x}{2}=0.$ Note that $N$ has fixed point, at the root $0$ of $D.$ We compute $D'(0)=2,$ and we see that $N'(0)=0.$ Thus the fixed point $0$ of $N$ is attracting fixed point. Obviously, orbit of any real point under $N$ converges to fixed point $0.$

(2) The Newton iteration function associated to $D(x)$ is $N(x)=x-\dfrac{2x-1}{2}=\dfrac{1}{2}.$ $N$ has fixed point, at the root $1/2$ of $D.$ We compute $D'(1/2)=2,$ and we see that $N'(1/2)=0.$ Thus the fixed point $1/2$ of $N$ is attracting fixed point. Obviously, orbit of any real point under $N$ converges to fixed point $\dfrac{1}{2}$ [8].

### *5.3 Sensitivity to Numerical Inaccuracies of Doubling Map*

The Doubling Map $D(x)=\begin{cases} 2x\,; & 0\le x<0.5 \\ 2x-1\,; & 0.5\le \mathrm{x}<1 \end{cases}$ is very sensitive to numerical inaccuracies. To see this, we calculate 100 values from the map, the first by using normal decimal numbers and then by using high-precision numbers. In the latter case, we start with numbers that have a precision of 65 digits:

```
vals1 = NestList[Piecewise[{{2#,0 ≤ # < 1/2}, {2# − 1, 1/2 ≤ # < 1}}]&,0.003,100];
```

```
vals2 = NestList[Piecewise[{{2#,0 ≤ # < 1/2}, {2# − 1, 1/2 ≤ # < 1}}]&,0.0003`65,100];
```

Values corresponding to vals2 are thick. From approximately iteration 50 on, the values differ greatly. In calculating vals2, we started with numbers having 65 digits of precision. During the calculation, many digits were lost so that the last value 0.6128only has a precision of approximately 38.20719793550678.Thus we know that all the digits of vals2 are correct. This means that the values in vals1are incorrect from approximately iteration 50 on. This demonstrates the sensitivity to numerical inaccuracies of the doubling map. Thus, if we calculate long sequences from the doubling map, it is important to use a high enough precision during the calculation. From the plot of vals2, we see that the series behaves quite chaotically. It is known that *chaotic* models are very sensitive to numerical inaccuracies.

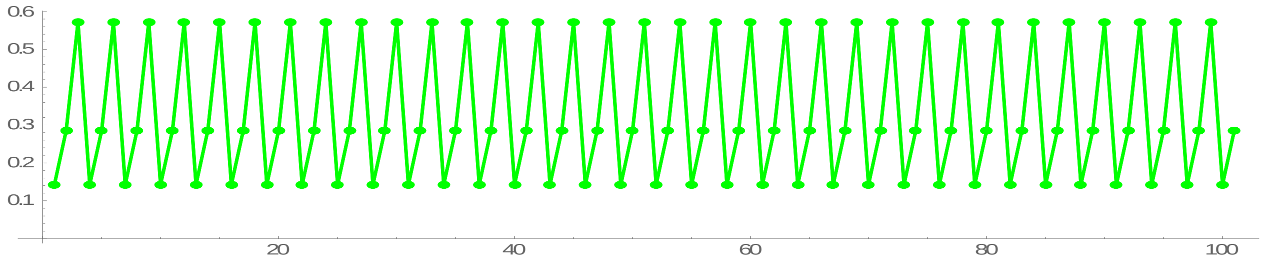



**Figure 3:** The initial x=1/7 and 0.1/7 `65, n=100

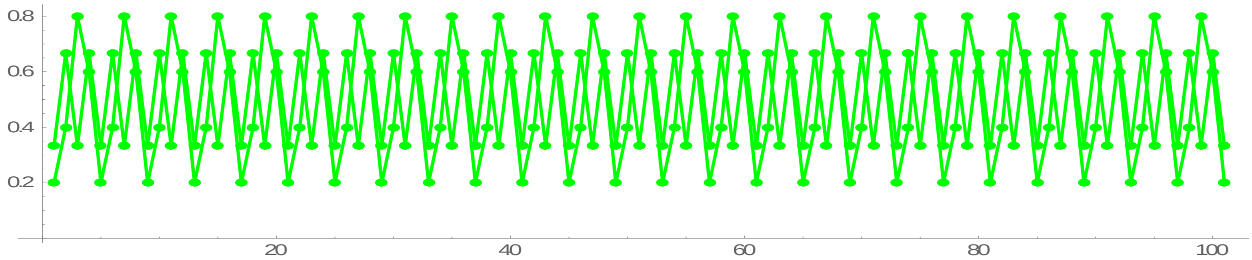



**Figure 4:** The initial x=1/3 and 0.1/5`65, n=100

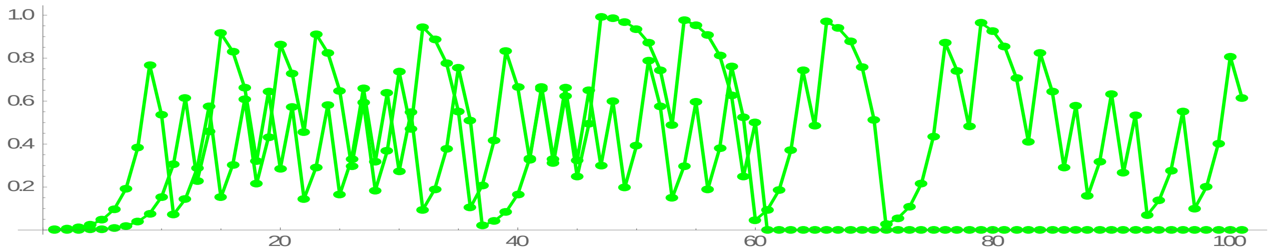


**Figure 5:** The initial x=0.003 and 0.0003`65, n=100

We see that Figure 3 and Figure 4 are regular which means that Numerical Accuracy. But the Figure 5 shows that the behavior is irregular which means Numerical Inaccuracies. In other words, it is called chaotic behavior of Doubling Map [5].

***5.4 Sensitivity Analysis to Initial Value of Doubling Map***

Chaotic models are also very sensitive to the initial value. To show this, compute, 50 iterations using starting points $0.02 + 10^{-i}$, $i$ = 1, …, 25. Then plot the 20th value of each of the 25 series. Also plot the 50th value of each of the 25 series:

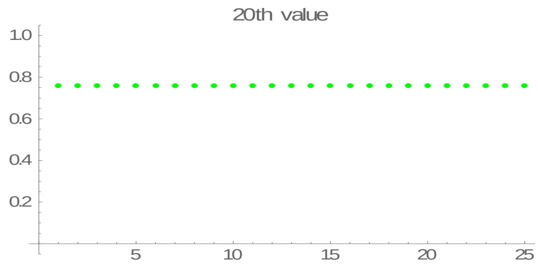



**Figure 6:** The initial x=0.02, n=50

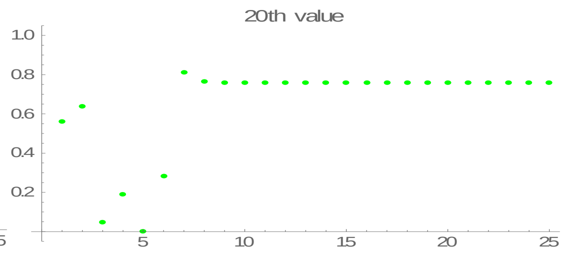



**Figure 7:** The initial x=0.02`50, n=50

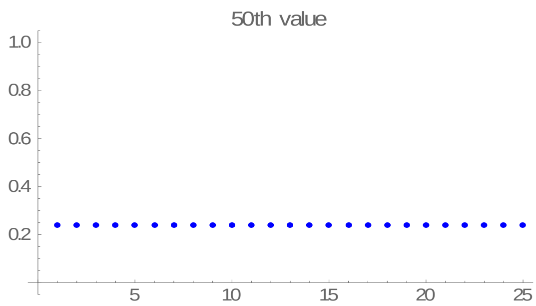



**Figure 8:** The initial x=0.02, n=50

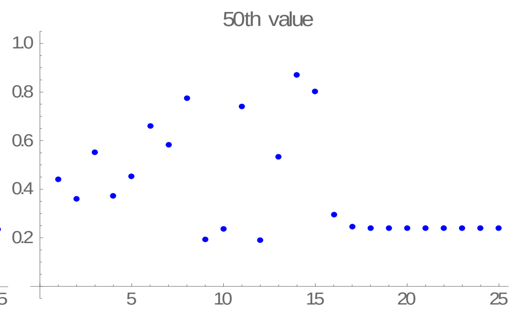



**Figure 9:** The initial x=0.02`50, n=50

From the first plot Figure 6 & 7, we see that even if the starting point differs from 0.02 by $10^{-7}$ or more (see the first seven points in the plot), the value of $D_{20}$ significantly differs from the value that results when starting from 0.02. From the second plot Figure 8 & 9, we see that if the starting point differs from 0.02 by $10^{-16}$ or more, the value of $D_{50}$ differs significantly from the value that result when starting from 0.02 [5].

### *5.5 Trajectories of Doubling Map*

We first calculate a solution set by starting from various points and iterating the equation *n*times. The starting points are chosen between x01and x02in steps of dx0. We get the following trajectories:

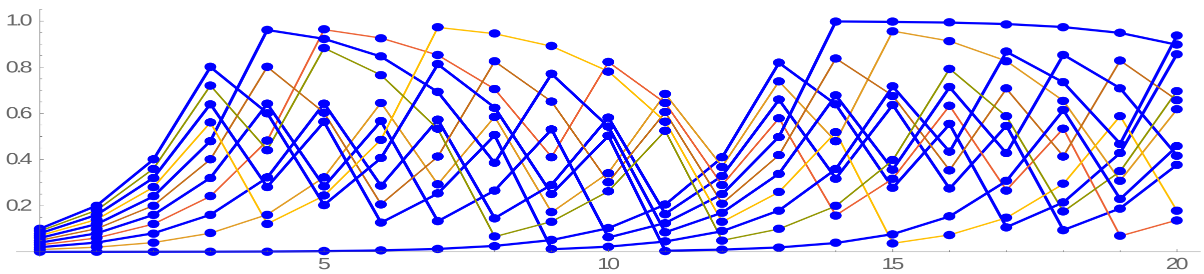



**Figure 10:** Trajectory for x01 =0.0001, x02=0.11, step size dx0=0.01, n=20.

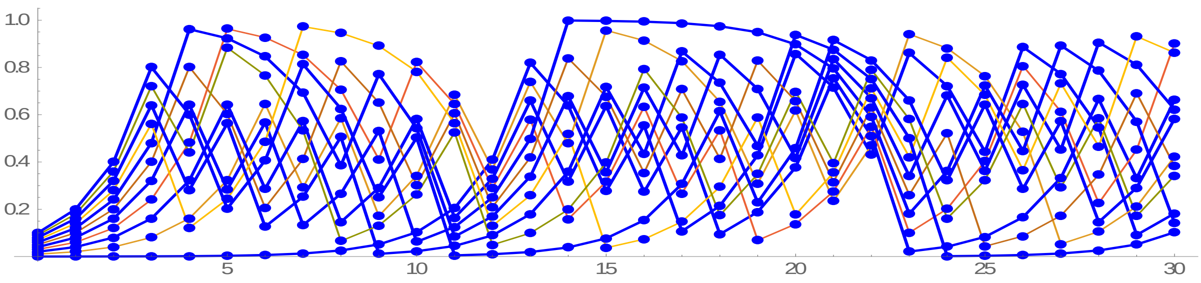



**Figure 11:** Trajectory for x01 =0.0001, x02=0.11, step size dx0=0.01, n=30.

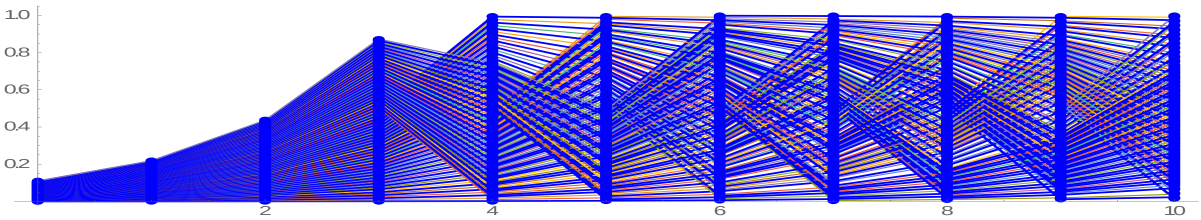



**Figure 12:** Trajectory for x01 =0.0001, x02=0.11, step size dx0=0.001, n=10.

We observe that if we take the starting values x01 and x02 are same even step size also same (10 and 11 figure) then the trajectories turns to chaotic with the increasing the number of iteration. If the step size is very small, the trajectories also chaotic in the Figure 12, [5].

### *5.6 Orbit Analysis of Doubling Map Graphically*

The Doubling Map is defined by

$$x_{n+1} = D(x_n) = \begin{cases} 2x_n & \text{for} \quad 0 \le x_n < \dfrac{1}{2} \\ 2x_n\text{-}1 & \text{for} \quad \dfrac{1}{2} \le x_n < 1 \end{cases}$$

Taking initial seeds from $[0,1)$ such that $(i)\ x_0 = \dfrac{1}{3}, (ii)\ x_0 = \dfrac{1}{5}, (iii)\ x_0 = \dfrac{1}{7}, (iv)\ x_0 = \dfrac{1}{11}, (v) = \dfrac{11}{13}$

$(vi)\dfrac{13}{29}, (vii)\dfrac{23}{59}, (viii)\dfrac{37}{69}$.

$(i)$ For $x_0 = \dfrac{1}{3}$, then $\left\{\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3}, \ldots, \dfrac{2}{3}, \ldots\right\}$;

$(ii)$ For $x_0 = \dfrac{1}{5}$, then $\left\{\dfrac{1}{5}, \dfrac{2}{5}, \dfrac{4}{5}, \dfrac{2}{5}, \dfrac{4}{5}, \ldots, \dfrac{2}{5}, \dfrac{4}{5} \ldots\right\}$;

$(iii)$ For $x_0 = \dfrac{1}{7}$, then $\left\{\dfrac{1}{7}, \dfrac{2}{7}, \dfrac{4}{7}, \dfrac{6}{7}, \dfrac{2}{7}, \dfrac{4}{7}, \dfrac{6}{7}, \ldots, \dfrac{2}{7}, \dfrac{4}{7}, \dfrac{6}{7}, \ldots\right\}$;

$(iv)$ For $x_0 = \frac{1}{11}$, then $\left\{\frac{1}{11},\frac{2}{11},\frac{4}{11},\frac{8}{11},\frac{6}{11},\frac{10}{11},\frac{2}{11},\ldots,\frac{2}{11},\frac{2}{11},\frac{4}{11},\frac{8}{11},\frac{6}{11},\frac{10}{11},\ldots\right\}$.

$(v)$ For $x_0 = \frac{11}{13}$, then $\left\{\frac{11}{13},\frac{9}{13},\frac{5}{13},\frac{10}{13},\frac{7}{13},\frac{1}{13},\frac{2}{13},\frac{4}{13},\frac{8}{13},\frac{3}{13},\frac{6}{13},\frac{12}{13},\frac{11}{13},\ldots\right\}$.

$(vi)$ For $x_0 = \frac{13}{29}$, then $\left\{\frac{13}{29},\frac{26}{29},\frac{23}{29},\frac{17}{29},\frac{5}{29},\frac{10}{29},\frac{20}{29},\frac{11}{29},\frac{22}{29},\frac{15}{29},\frac{1}{29},\frac{2}{29},\ldots\right\}$.

$(vii)$ For $x_0 = \frac{23}{59}$, then $\left\{\frac{23}{59},\frac{46}{59},\frac{33}{59},\frac{7}{59},\frac{14}{59},\frac{28}{59},\frac{56}{59},\frac{53}{59},\frac{47}{59},\frac{35}{59},\frac{11}{59},\frac{22}{59},\frac{44}{59},\ldots\right\}$.

$(viii)$ For $x_0 = \frac{37}{69}$, then $\left\{\frac{37}{69},\frac{5}{69},\frac{10}{69},\frac{20}{69},\frac{40}{69},\frac{11}{69},\frac{22}{69},\frac{44}{69},\frac{19}{69},\frac{38}{59},\frac{7}{69},\frac{14}{69},\frac{28}{69},\ldots\right\}$.

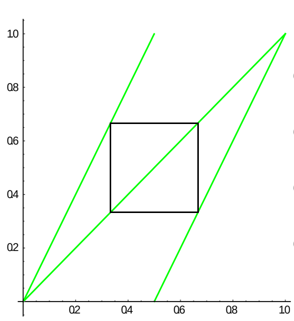


**Figure 13:** $x_0$ =1/3, n=1000

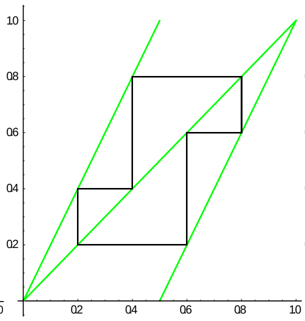


**Figure 14:** $x_0$ =1/5, n=1000

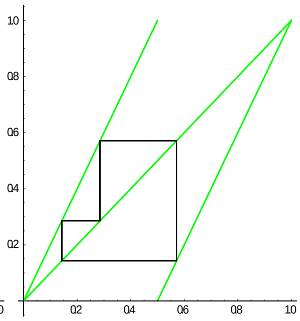


**Figure 15:** $x_0$ =1/7, n=1000

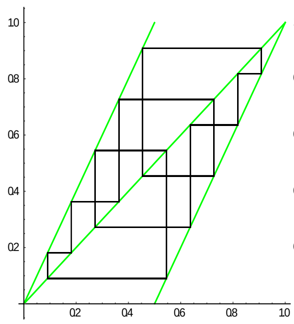


**Figure16:** $x_0$ =1/11, n=1000

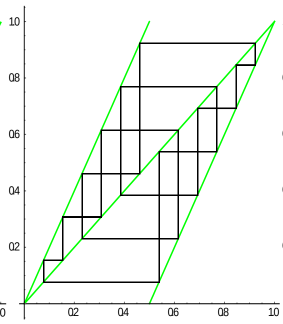


**Figure17:** $x_0$ =11/13, n=1000

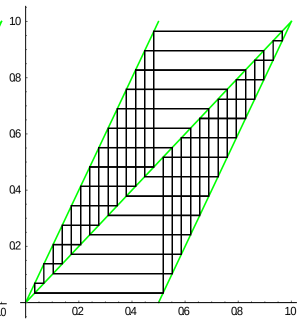


**Figure18:** $x_0$ =13/29, n=1000

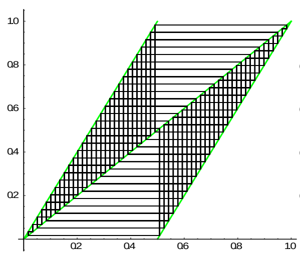


**Figure19:** $x_0$ =23/59, n=1000

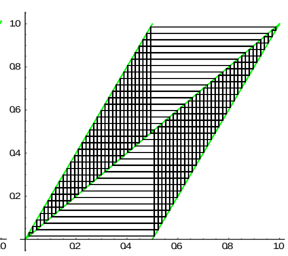


**Figure 20:** $x_0$ =37/69, n=1000

The sequence behave as follows: (i) there is period one behavior, (ii) there is period two behavior, (iii) there is a period three sequence, (iv) there is a period five sequence, (v) there is period twelve and in case (vi), (vii), (viii) are contained some list of iteration, but other method need to be used to established the long term behaviour of the sequences [2].

### *5.7 Staircase Diagram*

Again if we iterate the Doubling Map taking nearby initial seeds from $[0,1)$ graphically, we can get the following graph:

$(a_1)\,x_0 = N[1/\text{GoldenRatio}, 20]$, $(a_2)\,x_0 = N[1/\text{GoldenRatio}, 25]$, $(b_1)\,x_0 = N[1/\text{GoldenRatio}, 50]$, $(b_2)\,x_0 = N[1/\text{GoldenRatio}, 51]$, $(c_1)\,x_0 = N[1/\text{GoldenRatio}, 200]$, $(c_2)\,x_0 = N[1/\text{GoldenRatio}, 250]$, $(e_1)\,x_0 = N[1/\text{GoldenRatio}]$, $(e_2)\,x_0 = N[1/\text{GoldenRatio}, 320]$, $(f)\ x_0 = N[1/\text{GoldenRatio}, 500]$.

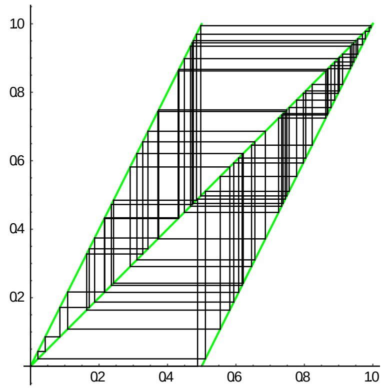


**Figure 21:** For $(a_1)\,x_0 = N[1/\text{GoldenRatio}, 20]$

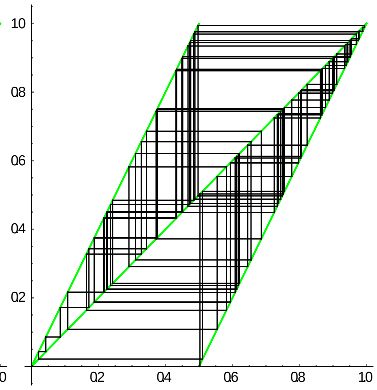


**Figure 22:** For $(a_2)\,x_0 = N[1/\text{GoldenRatio}, 25]$

Difference: $0.\times 10^{-21}$

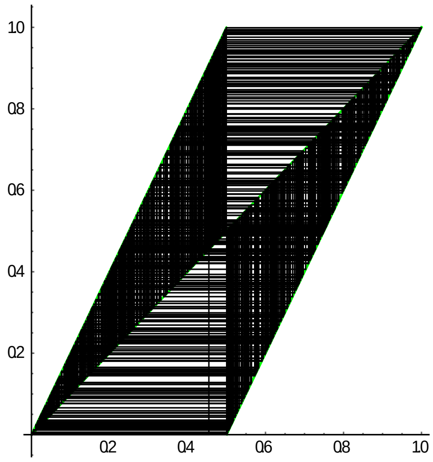


**Figure 23:** For $(c_1)\,x_0 = N[1/\text{GoldenRatio}, 200]$

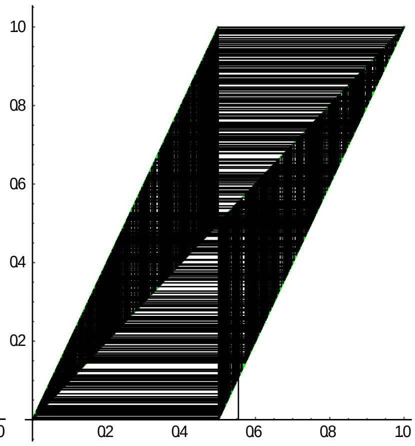
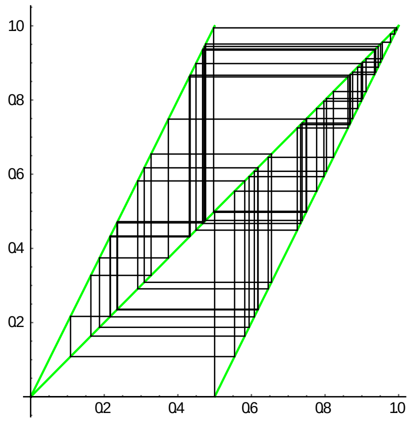


**Figure 24:** For $(c_2)\,x_0 = N[1/\text{GoldenRatio}, 250]$

Difference: $0.\times 10^{-201}$



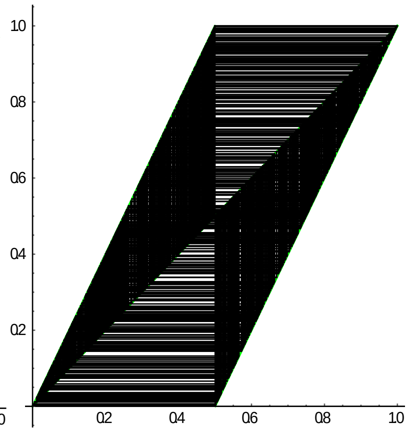




**Figure 25:** For $(e_1)\, x_0 = N[1/\text{GoldenRatio}]$ **Figure 26:** For $(d_2)\, x_0 = N[1/\text{GoldenRatio}, 302]$

$\text{Difference}: 1.110223024625156 \times 10^{-16}$

Each of the diagrams (Figure13-26) can be reproduced by using Mathematica. Figure show that the doubling map displays sensitivity to initial conditions and can be described as being chaotic. The iterative path plotted in Figure 23, 24, 26 appears to wander randomly. It clearly shows the sensitivity to initial conditions. It is clear from the diagrams is that the three basic properties of chaos and they are mixing, periodicity, sensitivity to initial conditions, and they are all exhibited for certain values. Indeed, a now famous result due to Li and [3] states that if a system displays period-three behaviour, then the system can display periodic behaviour of any period. Li and Yorke then go on to prove that the system can display chaotic phenomena [2].

## 6. Conclusion

Chaotic behavior of one dimensional Doubling map introduces an interesting and exciting part of Dynamical Systems. In this paper, we have introduced one dimensional Doubling map which is shown as chaotic map in several senses by some pioneers. We are trying to find the dynamical behaviors of others one dimensional maps with applications and establish some mathematical formulas concerning chaotic dynamical systems.


## Acknowledgement

It is starting with thanking the Almighty Allah. Not only this paper but also my whole life is blessed by the grace of Him. I acknowledge with a humble respect to my supervisor **Professor Dr. Payer Ahmed**, Department of Mathematics, Faculty of Science, Jagannath University, Dhaka and **Professor Dr. Md. Rezaul Karim**, Chairman, Department of Mathematics, Jagannath University, Dhaka for delivering right directions to complete this research work properly. I am also gratitude to my teacher **A.B.S. Manik Munsi**, Lecturer, Department of Mathematics, Jagannath University, Dhaka for his constructive guides during my research works.


## 7. Constraints/Limitations of the study

There are some limitations we see in our article. First of all we have to face software problem while preparing

Orbit analysis and analyzing staircase diagram of the Doubling map. First we tried to see the said analysis using Mathematica 5.0 version but it did not working showing graph. After that we took an attempt to use Mathematica 11.0 version. Then we have seen that it worked very smoothly and got our expected result. Therefore Software version problem can be treated as our major limitation of our paper. The 2$^{nd}$ limitation is that as the Doubling map does not come any parameter so we did not have desired chaotic sound like others chaotic maps such as Logistic map and Tent map. Of course high speed super computer is very much needed to run the Doubling map related program to run quickly.

## 8. Recommendations

We would like to notice here that Mathematica 11.0 version and some other upgraded versions are recommended to use while performing graphical Analysis, Orbit analysis, Staircase diagram etc. To run the program more smoothly and to find actual chaotic sound a powerful high speed super computer are recommended for such a research so that the ambitious researchers can get their expected results.

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